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Propagation of Electromagnetic Waves in a Random Medium and Nonzero Rest Mass of the Photon

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The propagation of electromagnetic waves is studied in a Maxwell vacuum with $\sigma \neq 0$. The photon loses energy during its propagation through this vacuum. This dissipation of energy is related to the fluctuation of the refractive index of the underlying vacuum. There exists a bounded and unique solution in the limit $\sigma \to 0$ in the asymptotic region. The geometric structure of the background space-time is Finslerian in nature.

1. INTRODUCTION

Maxwell's equations have been studied (Vigier, 1990) in a vacuum with nonzero conductivity coefficient, i.e., with $\sigma \neq 0$. The nonzero conductivity coefficient gives rise to a dissipative term in the field equation. In this case if we consider the propagation of a photon through this type of vacuum, the photon acquires a mass at cosmological scale (Fulli, 1975; de Broglie and Vigier, 1972; Marochnik, 1968; Kar *et al.*, 1993). In fact, due to the presence of the dissipative term in the field equation, the photon loses energy during the propagation through this vacuum. This dissipation can be related to the fluctuation of the refractive index of the underlying vacuum. In this paper we study the wave equation with random refractive index and show that there exists a bounded and unique solution of this wave equation for small σ at $r \to \infty$. This is consistent with the conclusion drawn by Vigier (1990). In this model of a fluctuating vacuum, the velocity of propagation of the disturbance, i.e., the phase velocity, is shown to be finite and no superluminal transmission is allowed. In Section 2 we briefly discuss the Maxwell equations

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in vacuo with $\sigma \neq 0$. This gives rise to a Finslerian structure of the vacuum as considered by Synge (1966). This is discussed in Section 3.

2. MAXWELL'S EQUATIONS *IN VACUO* WITH $\sigma \neq 0$

If we endow the vacuum with nonzero conductivity coefficient $\sigma \neq 0$, Maxwell's equations can be written in the form

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0 \\ \operatorname{curl} \mathbf{H} &= \sigma \mathbf{E} + \epsilon_0 \chi_e \frac{\partial \mathbf{E}}{\partial t} \\ \operatorname{div} \mathbf{H} &= 0 \\ \operatorname{curl} \mathbf{E} &= -\mu_0 \chi_m \frac{\partial \mathbf{H}}{\partial t} \end{aligned} \tag{1}$$

where ϵ_0 denotes the vacuum's dielectric constant, μ_0 denotes the vacuum's permeability constant, χ_e is the relative dielectric constant, and χ_m is the relative permeability constant. Again,

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E}$$

so,

$$\nabla^2 \mathbf{E} = -\frac{\epsilon_0 \chi_e \chi_m}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \mu_0 \chi_m \frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

If we consider plane waves in the z direction, i.e.,

$$\begin{aligned} E_x &= b \exp\left[i\omega\left(t - \frac{z}{v} \right) \right] \\ H_y &= b \left(\frac{\epsilon_0 \chi_e}{\mu_0 \chi_m} \right)^{1/2} \exp\left[i\omega\left(t - \frac{z}{v} \right) \right] \end{aligned} \tag{3}$$

then, putting $q = 1/v$ in the plane-wave solution of $\mathbf{E}$ in equation (2), we get

$$q^2 = \frac{\chi_e \chi_m}{c^2} \left(1 - \frac{i\sigma}{\epsilon_0 \chi_e \omega} \right) \epsilon_0 \mu_0 \tag{4}$$

Here, q can be considered as complex in nature, having the form $\alpha - i\beta$, where α and β are real and given by

$$\alpha = 1 + \frac{1}{8}\left(\frac{\sigma^2}{\epsilon_0^2\chi_e^2}\right)\frac{1}{\omega^2} + O\left(\frac{\sigma^4}{\omega^4}\right) \tag{5}$$

$$\beta^2 = \frac{1}{2}\frac{\sigma^2}{\epsilon_0^2\chi_e^2}\frac{1}{\omega^2} \tag{6}$$

for $(\sigma/\omega) \to 0$.

Then the velocity defined by v in equation (4) will give rise to a complex refractive index η in the vacuum. The velocity $v = 1/\alpha$ is the phase velocity of propagation of the disturbance through the underlying vacuum. Henceforth, it will be denoted as v_P. After a simple calculation (Fulli, 1975; de Broglie and Vigier, 1972; Marochnik, 1968; Kar *et al.*, 1993), the phase velocity can be written as

$$v_p = \eta\left(1 - \frac{1}{8}\frac{\sigma^2}{(\epsilon_0\chi_e)^2\eta^4\omega^2}\right) \tag{7}$$

and the group velocity as

$$v_g = \eta\left(1 + \frac{1}{4}\frac{\sigma^2}{\epsilon_0^2\chi_e^2\eta^4\omega^2}\right)^{1/2} \tag{8}$$

It is clear from the relation (7) that the phase velocity will be finite and less than c (for $\eta < 1$). So, the velocity of propagation of the disturbance will be finite and no superluminal transmission is allowed in this vacuum. This might play a significant role in quantum mechanics.

Taking the above calculated values of α and β in E_x and H_y, the two following cases may arise:

(i) Plane waves are progressively damped with the decay factor $\exp(-kz)$, where $k = \omega\beta$.

(ii) The velocity of propagation of the wave is given by v_g and it varies with the frequency.

Using the de Broglie relation

$$E = \frac{m_\gamma c^2}{(1 - c^2)^{1/2}}$$

we get

$$m_\gamma^2 = h^2\omega^2(1 - \eta^2) - \frac{h^2}{4}\frac{\sigma^2}{(\epsilon_0\chi_e)^2\eta^2} \tag{9}$$

For complex refractive index, say $\eta^2 = -1$, and at low frequency,

$$m_\gamma \simeq \frac{h\sigma}{2(\epsilon_0 \chi_e)\eta} \simeq \frac{hH}{2c^2} = 10^{-65}\ \text{g} \tag{10}$$

where H is the Hubble constant.

Vigier got exactly the same mass m_γ as we have in (10). But, in general, m_γ depends on the frequency, as is evident from (9). It is now clear from the above analysis that the progressive waves will be heavily damped for complex refractive index of the vacuum and we get the nonzero rest mass of the photon. This dissipation can be related to the fluctuation of the medium, which can be characterized by a random refractive index. It can be envisaged in the following way. Starting with equation (2) and taking the time-dependent part of **E**, we get

$$\nabla^2 \mathbf{E} + (\chi^2 k_0^2 \chi_m \chi_e + i\mu_0 \chi_m \sigma k_0)\mathbf{E} = 0 \tag{11}$$

Now,

$$\eta = (\mu_0 \epsilon_0)^{1/2}$$

or

$$\mu_0 = \eta^2 / \epsilon_0$$

Then equation (11) becomes

$$\nabla^2 \mathbf{E} + \eta^2 k_0^2 \chi_m \chi_e \mathbf{E} + i\eta^2 \frac{\chi_m k_0 \sigma}{\epsilon_0} \mathbf{E} = 0 \tag{12}$$

Here, η^2 is supposed to be a random function. Let $\eta^2 = 1 + \mu(\omega, r)$; then equation (12) gives

$$\nabla^2 \mathbf{E} + (1 + \mu) k_0^2 \chi_m \chi_e \mathbf{E} + i\frac{(1 + \mu)\chi_m \sigma k_0}{\epsilon_0} \mathbf{E} = 0 \tag{13}$$

Equation (13) is a random differential equation having the random coefficient of **E**. Several authors have already studied the wave equation with random refractive index (Frisch, 1964) in the form

$$\nabla^2 \psi + k_0^2 (1 + \mu(\omega, r))\psi = 0 \tag{14}$$

where $\eta^2 = 1 + \mu(\omega, r)$ is the refractive index. By replacing k_0 by $k_0 + i\eta$ ($\eta > 0$), it has been shown (Frisch, 1964) that equation (14) has bounded and unique solution in the asymptotic region (i.e., $r \to \infty$). This holds for small values of η, i.e., $\eta \to 0$. Then equation (14) can be written as

$$\nabla^2 \psi + k_0^2 (1 + \mu)\psi + 2i\eta k_0 (1 + \mu)\psi = 0 \tag{15}$$

Now comparing equation (13) with (15), we can write

$$\eta \simeq \chi_m \sigma / \epsilon_0 \tag{16}$$

Since the bounded solution exists for $\eta \to 0$, σ should be very small. Hence, equation (13) has a bounded and unique solution in the asymptotic region for $\sigma \to 0$. In fact, the introduction of $k_0 + i\eta$ instead of k_0 in equation (14) implies the introduction of small dissipation during the propagation of a wave in a medium with random refractive index. So we can associate the dissipation of the energy of a photon in a Maxwell vacuum ($\sigma \neq 0$) with a fluctuating vacuum which can be characterized by a random refractive index.

3. FINSLERIAN STRUCTURE OF VACUUM

If the vacuum is endowed with random refractive index η, then we get the following dispersion relation in a covariant form:

$$[|\mathbf{k}|^2 - \eta^2 k_0^2] A^\mu(k) = \frac{\mu}{c}\left(g^{\mu\nu} + \frac{\eta^2 - 1}{\eta^2} u^\mu u^\nu\right) J_\nu(k) \tag{17}$$

with

$$k = (\mathbf{k}, k_0), \qquad J^\mu = (0, \sigma \mathbf{E})$$

$$A^\mu = \left(\mathbf{A}, i\frac{\phi}{c}\right)$$

and $u = (\mathbf{0}, 1)$, which is unit timelike vector denoting the velocity of the medium. σ denotes the conductivity of the medium. It is evident from (17) that $|\mathbf{A}| \neq 0$, but $\phi = 0$ for $\sigma \neq 0$, which is nothing but the usual Coulomb gauge. The condition $A_\mu A_\mu = 0$ is not consistent with the usual Coulomb gauge. So it seems that the gauge principle has to be reinterpreted for $m_\gamma \neq 0$ with $\sigma \neq 0$ in the vacuum. It has been shown (Roy, n.d.) that the nonzero rest mass of the photon with $\sigma \neq 0$ is consistent with gauge invariance of the first and second kind if we introduce the fourth component of the current as $J_0 \sim B_0$ (instead of zero), where B_0 is the magnetic flux density associated with a single photon (Evans, 1994). From (17) we construct an effective metric tensor of the background metric tensor $g^{\mu\nu}$ as

$$G^{\mu\nu} = g^{\mu\nu} + \frac{\eta^2 - 1}{\eta^2} u^\mu u^\nu \tag{18}$$

By taking the average of the random refractive index, we obtain the average metric tensor

$$\overline{G}^{\mu\nu} = g^{\mu\nu} + \frac{m^2 - 1}{m^2} u^\mu u^\nu \tag{19}$$

where $\langle 1/\eta^2 \rangle = 1/m^2$, $\langle , \rangle$ being the statistical average.

This form of the metric tensor $\overline{G}_{\mu\nu}$ has been discussed by Synge (1966) in studying the gravitational and electromagnetic fields on the generalized Lagrange space in dispersive as well as in nondispersive media from the point of view of relativistic geometrical optics. Synge considered a generalized metric tensor $g_{\mu\nu}$ as

$$g_{\mu\nu}(x, V(x)) = \Gamma_{\mu\nu}(x) + \left(1 - \frac{1}{n^2(x, V(x))}\right) V_\mu V_\nu \tag{20}$$

where $V^i(x)$ is the velocity field of the medium.

The medium $\mu = [M, V(x), n(x, V(x))]$ is called a dispersive medium where M is the manifold and $n(x, V(x))$ is the refractive index. If $\partial n/\partial V = 0$, then μ is called a nondispersive medium. If $1/n^2 = 1 - (1/c^2)$, then $g_{\mu\nu}(x, V(x))$ is reduced to the metric $(\Gamma_{\mu\nu} + (1/c^2)Y_\mu Y_\nu)$, which was considered by Miron and Kawaguchi (1991). It is easy to check that

$$g_{\mu\nu}(x, V)g^{\nu\xi}(x, V) = \delta^\xi_\mu \tag{21}$$

In our framework, the metric tensor constructed by averaging over the refractive index should be the metric tensor for a nondispersive medium since $\partial m/\partial u = 0$. So, the metric of the Maxwell vacuum with $\sigma \neq 0$ should be Finslerian in nature. If $n = 0$, it reduces to the Riemannian structure. For $n = 1$ we get the usual Maxwell vacuum where the photon does not lose energy during its propagation.

4. CONCLUSION

It is evident from the above analysis that a geometrical structure of the Maxwell vacuum with $\sigma \neq 0$ can be constructed which gives rise to nonzero mass of the photon as it propagates through the vacuum. This may give rise to new insights regarding the interpretation of the anomalous redshift at the cosmological scale. These issues will be studied in subsequent publications.

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